Snowflakes Among Tropical Trees

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Outline

Tropical Beaches
- The Tropical World
- Linear Spaces

Tropical Forest
- Tropicalization
- Phylogenetic Trees

Tropical Snowflakes
- Climbing the Trees
- Culmination

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Culmination

Tropical Semiring

- The *tropical semiring* is the set $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$ equipped with two operations:
  - $\oplus = \min$,
  - $\otimes = +$.
- For example, $4 \oplus 9 = 4, 4 \otimes 9 = 13$.

Tropicalization

- We can *tropicalize* a polynomial by replacing addition and multiplication with their tropical counterparts.
- For example,

$$f = x^2 + y^2 - 1$$

  becomes

$$F = \min(2x, 2y, 1).$$

- Then the associated *tropical hypersurface* is where F is non-differentiable.

Example

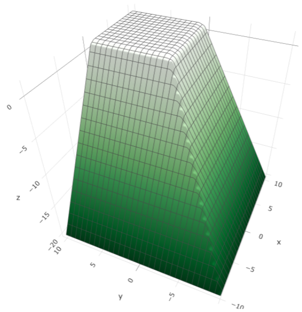


Figure: A graph of $z = \min(2x, 2y, 1)$

Example

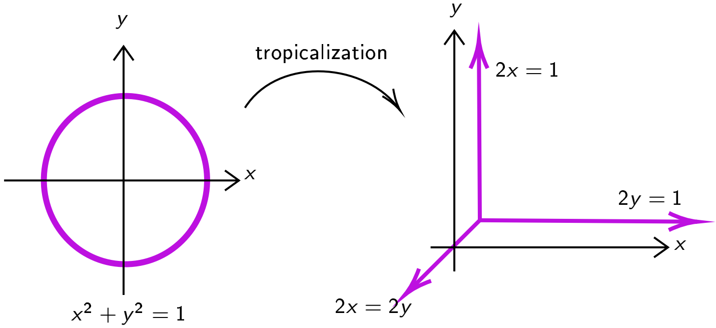


Figure: Tropicalization of a circle

Outline

The Grassmannian

- We define the Grassmannian $\mathrm{Gr}(k, n)$ as the set of all k-dimensional linear subspaces of an n-dimensional vector space $V = K^n$ over a field K.

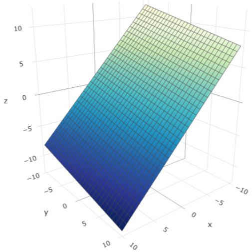


Figure: A 2D plane in 3D space

The Stiefel Map

- The *Stiefel map* takes a $k \times n$ matrix A and maps it to its row space.
- If A is of full rank, then this row space is k-dimensional, and is a point in $\mathrm{Gr}(k, n)$.

The Plücker Embedding

- We can embed the Grassmannian into projective space $\mathbb{P}\left(\Lambda^k V\right)$, with $\Lambda^k V$ the k-th exterior power of V.
- This embedding is realized with *Plücker coordinates* $p_{i_1,i_2,\ldots,i_k}$, and live in $\binom{n}{k}$-dimensional projective space.

The Plücker Relations

- $\mathrm{Gr}(k, n)$ can be realized as an intersection of quadric equations in Plücker coordinates called the *Plücker relations*.
- For the case of $\mathrm{Gr}(2, n)$, we have that

$$p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk} = 0 \tag{1}$$

for $1 \leq i < j < k < l \leq n$.

Outline

The Tropical Grassmannian

- We can tropicalize the Plücker relations to get the *tropical Grassmannian* $\operatorname{tropGr}(k, n)$
- For the case of $\operatorname{tropGr}(2, n)$, we have that

$$p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk} = 0 \tag{2}$$

becomes the statement that

$$\min(p_{ij} + p_{kl}, p_{ik} + p_{jl}, p_{il} + p_{jk}) \tag{3}$$

is achieved twice.

The Tropical Stiefel Map

- We can also get the *tropical Stiefel map* $\pi(A)$.
- For the $k = 2$ case, we have that

$$[\pi(A)]_{ij} = \min(a_{1i} + a_{2j}, a_{2i} + a_{1j}) \tag{4}$$

- However, we find that it is not surjective, as it is in the classical case.

The Goal

Our goal is then to find a point in the tropical Grassmannian which does not lie in the tropical Stiefel image.

Outline

Phylogenetic Trees

- A *phylogenetic tree* is a tree with n labeled leaves and no vertices of degree 2.

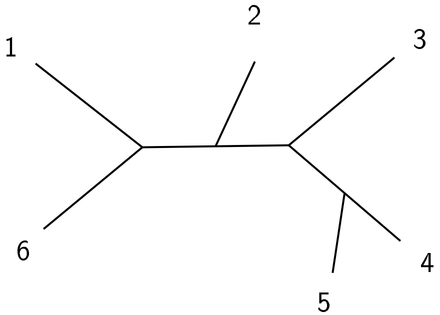
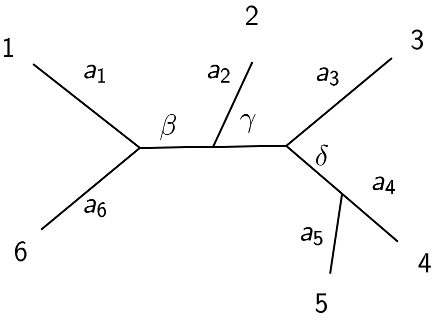


Figure: A phylogenetic tree with 6 leaves.

Tree Metric

- A *tree metric* is a finite metric d_{ij} on the set $[n] = 1, 2, \dots n$ arising from distances on a tree.
- When is a metric a tree metric?

Example



$$d_{12} = a_1 + \beta + a_2$$
$$d_{34} = a_3 + \delta + a_4$$
$$d_{56} = a_5 + \delta + \gamma + \beta + a_6$$
$$\vdots$$

Figure: A metric built from edge weights.

Four Point Condition

Theorem (Four Point Condition)

A metric d_{ij} on the set $[n]$ is a tree metric iff

$$\max(d_{ij} + d_{kl}, d_{ik} + d_{jl}, d_{il} + d_{jk}) \tag{5}$$

is achieved twice.

Comparison

- Compare the four point condition

$$\max(d_{ij} + d_{kl}, d_{ik} + d_{jl}, d_{il} + d_{jk})$$

to the Plücker relations

$$\min(p_{ij} + p_{kl}, p_{ik} + p_{jl}, p_{il} + p_{jk}).$$

Connection to tropGr$(2, n)$

- tropGr$(2, n)$ is the space of phylogenetic trees.
- So, to find an element of tropGr$(2, n)$ outside the Stiefel image, we want to find a tree that does not arise from the latter.

Outline

The Caterpillar Tree

- For a tree in the Stiefel image, it is shown that the bounded part of the tree is homeomorphic to a line segment.
- This makes it a caterpillar tree: every leaf is branching off from one central path.

Example

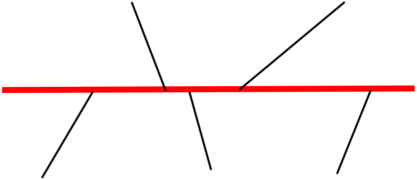


Figure: A caterpillar tree (with the central path in red).

Outline

The Snowflake Tree

- It then follows that a tree outside of the Stiefel image is one which is not a caterpillar tree.
- The smallest such tree is called the *snowflake tree*, corresponding to a point in $\text{tropGr}(2, 6)$.

The Snowflake Tree (cont.)

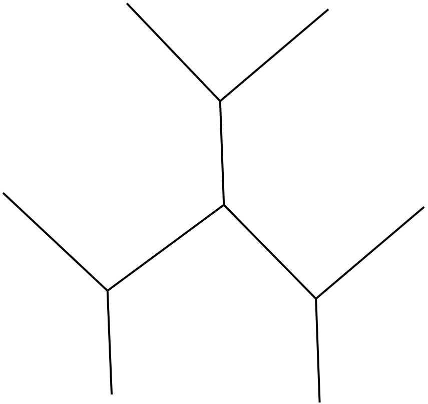


Figure: The snowflake tree.