Counting Faces with Shellability

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Outline

1. Preliminaries
   - Polytopes
   - Faces
2. Shellability
   - Complexes
   - Shellings
3. Dehn-Sommerville
   - f-vectors
   - h-vectors
   - Dehn-Sommerville Equations

Outline

What are Polytopes?

- $\mathcal{V}$-polytopes : convex hulls
- $\mathcal{H}$-polytopes : intersections of half-spaces
- A *polytope* can be presented in either fashion (non-trivial)

Simplices

- *d-simplex* : convex hull of $d + 1$ affinely independent points
- *standard d-simplex Δ_d* : convex hull of the $d + 1$ unit vectors in $\mathbb{R}^{d+1}$

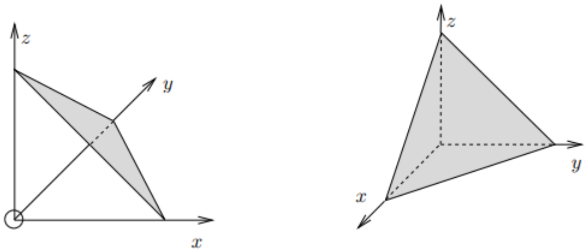

Figure: Example of a simplex: Δ_2 in $\mathbb{R}^3$

Outline

Faces

- An inequality of the form $\vec{a} \cdot \vec{x} \leq a_0$ is said to be *valid* if it is true for all $x \in P$ a polytope.
- Given a valid inequality, a *face* is the subset $P \cap \{\vec{x} : \vec{a} \cdot \vec{x} = a_0\}$, i.e. where *equality* holds.

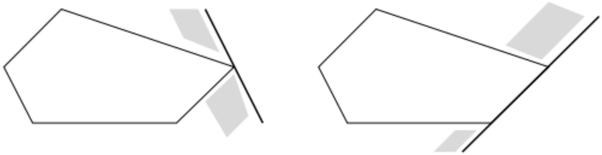

Figure: Examples of faces: a vertex and an edge

Euler's Polyhedron Formula

Theorem (Euler's Polyhedron Formula)

$$V - E + F = 2$$

where V, E, and F are the number of vertices (0-faces), edges (1-faces) and facets (2-faces) of a 3-polytope.

- Is there something similar in higher dimensions?

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Dehn-Sommerville Equations

Theorem (Dehn-Sommerville Equations)

The h-vector of the boundary of a simplicial d-polytope satisfies

$$h_k = h_{d-k}$$

where $h_k := \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{i-1}$

Outline

Complexes

- A *polytopal complex* $\mathcal{C}$: finite collection of polytopes
- A *pure complex* : all facets are the same dimension
- A *simplicial complex* : all faces are simplices
- A *boundary complex* $\mathcal{C}(\partial P)$ = the facets of P

Outline

Shellings

- A *shelling* of a pure polytopal complex is an ordering $F_1, F_2, \ldots F_s$ of its facets such that
  1. The boundary complex $\mathcal{C}(\partial F_1)$ has a shelling
  2. The intersection of the facet F_j with the union of the previous facets is the beginning of a shelling of $\mathcal{C}(\partial F_j)$

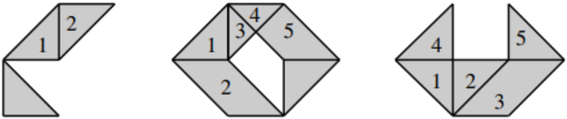

Figure: Some 2-complexes

Properties

- A shelling of a boundary complex of a polytope is reversible.
- Polytopes are shellable.

Outline

f-vectors

- Let f_k be the number of k-faces of a polytopal complex $\mathcal{C}$
- The *f-vector* of $\mathcal{C}$ is the vector $\vec{f} = (f_{-1}, f_0, \ldots, f_d)$.

Euler-Poincaré Formula

Theorem (Euler-Poincaré Formula)

$$f_0 - f_1 + \cdots + (-1)^{d-1} f_{d-1} = 1 - (-1)^d$$

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Motivation

- The h-vector arises from counting the number of parts of a given size of a partitionable simplicial complex
- Partitions can arise from shellings : h-vector independent of the shelling chosen

Defining h-vectors in general

Define the h-vector of a simplicial complex of dimension $d-1$ to be

$$\vec{h}(\mathcal{C}) = (h_0, h_1, \ldots, h_d)$$

where

$$h_k := \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{i-1}$$

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Dehn-Sommerville Equations

Theorem (Dehn-Sommerville Equations)

The h-vector of the boundary of a simplicial d-polytope satisfies

$$h_k = h_{d-k}$$

Proof (McMullen 1970)

- Shellings of boundary complexes of polytopes are reversible, so whatever F_j contributes to the original shelling at h_k, it also contributes to the reverse shelling at h_{d-k}.
- But the h-vector is independent of the shelling chosen, so $h_k = h_{d-k}$

Further Results

- When $k = 0$, we get the famous Euler-Poincaré formula.
- We can find an upper bound on the number of k-faces of a d-polytope with n vertices (McMullen 1970)